"BOCHNER INTEGRALS AND VECTOR MEASURES"

Project for the Degree of M.S.

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BOCHNER INTEGRALS

AND

VECTOR MEASURES

By
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ABSTRACT

This project extends known theorems for scalar valued functions to the context of Banach space valued functions. In particular, it contains generalizations of the classical theory of Lebesgue Integrals, complex measures, Radon-Nikodym theorem and Riesz Representation theorem. We explore some properties of functions whose domains are abstract Banach spaces, where the usual derivatives are replaced by Radon-Nikodym derivatives.

The first two Chapters are devoted to infinite dimensional measurable functions and the problem of integrating them. Most of the basic properties of Bochner integration are forced on it by the classical Lebesgue integration and the usual definition of measurability.

The Radon-Nikodym theorem for Bochner Integral is the subject to Chapter III. The roles of reflexive spaces, separable anti-dual spaces and the Radon-Nikodym property of Banach spaces are also discussed in this Chapter. One of the most interesting aspects of the theory of the Bochner integral centers about the following questions: When does a vector measure $F:\Im\longrightarrow X$ arise as a Bochner integral of an $L^1(S,X)$ function (i.e. $F(E)=\int_E f dm$)?

And conversely, if $f\in L^1(S,X)$. Then, is $F:\Im\longrightarrow X$, defined by $F(E)=\int_E f dm$, a countably additive vector measure, absolutely continuous with respect to the positive measure m? These two questiones are examined by the Radon-Nikodym theorem and the Riesz Representation theorem. It is worth observing, that the relationsip between these theorems are considered to be just a formality of translating a set of basic definitions from one context to another.

There are theories of integration similar to the Bochner Integral, that allow us to integrate functions that are only weakly measurable (The Pettis Integral) with respect to a positive measure. Also, the ultimate generallity of the Bochner Integral, the Bartle Integral, for integrating vector valued functions with respect to a general vector measure. However, these theories do not occupy a central role in our study and we limit ourselves to only mentioning [1] as an excellent reference.

CONTENTS

Section **Page**

INTRODUCTION

The theory presented in this report may be used in a variaty of ways. One application of it is in the study of the Radon-Nikodym theorem and its relations to the topological and geometric structure of Banach spaces. Another one concerns existance proofs in some infinite dimensional problems. Often, people obtain estimates on solutions to approximate problems in an $L^p(S,X)$ space and it is nice to be able to use that $L^p(S,X)$ is reflexive, provided that X is. Thus, applying the Eberlein-Smulian theorem, one can extract a weakly convergent subsequence, the limit of which will, sometimes, be a solution of the problem. This is a standard procedure used in books like [1], [9], [11] and prerequisite material to read many papers, eg. [2], [4], [8], [15], [16]. Next, but not less important, is the use of the Radon-Nikodym and the Riesz Representation theorems in the theory of integral representation of linear compact operators in $\mathcal{L}\left(L^1(S,\mathfrak{I},m)\ ;\ X\right)$, see [1].

Though, all of the facts, theorems and results in the project are based on "Vector Measures", by Diestel (Kent State U.) and Uhl (U. of Illinois), there are significant differences in the presentations, some of which we would like to point out:

1) The real Banach spaces and the dual spaces are extended in the project to complex spaces and anti-dual spaces, respectively. Note that Λ element of the anti-dual space X' means that $\Lambda(kx+y)=\bar{k}\Lambda(x)+\Lambda(y)$. One reason for using the anti-dual space rather then the dual space is that the Riesz map, $R:H\longrightarrow H'$, defined by $Rx(y)=(x,y)_H$, is linear, for H' being the anti-dual of the Hilbert space H;

2) About the definition of a measurable function: In the project: $x(\bullet):S\longrightarrow X$ is measurable, if $x_n(s)\xrightarrow[n\to\infty]{}x(s),\ \forall\ s\in S$. However, the book and most other sources, only require $x_n(s)\xrightarrow[n\to\infty]{a.e.}x(s)$. Here, $x_n(\bullet)$ are simple functions. Of course, both definitions are the same if the measure space $(S,\mathfrak{I},m)$ is complete (i.e. $A\subseteq B\subseteq C$, A, $C\in\mathfrak{I}$ and $m(C-A)=0$ implies $B\in\mathfrak{I}$).

For example, Lebesgue measure on $(-\infty, \infty)$. Our approach has the advantage of ensuring validity of the theorem that $x(\bullet)$ is measurable if and only if $x^{-1}(U)$ is measurable, whenever U is open, even in the case where $(S,\Im,m)$ is not complete (say Lebesgue measure on the σ-algebra of the Borel sets, or a product measure).

Having this theorem simplifies the presentation of the Pettis' Measurability theorem and harmonizes better with the standard theory for scalar valued functions, where the measurable functions are defined by saying that the inverse images of open sets are measurable, see [13];

3) In the book the Riesz Representation theorem is stated and proved as a necessary and sufficient condition (i.e. $\left(L^p(S,X)\right)' \cong L^{p'}(S,X') \Leftrightarrow X'$ has the Radon-Nikodym property with respect to the finite measure m). However, in this report we leave out the proof of necessity deliberately.

In this project the reader may find remarkable similarities between most of the results developed for functions with values in a Banach space and scalar valued case. For example, the proofs of Radon-Nikodym theorem, Riesz Representation theorem and reflexiveness of $L^p(S,X)$, in the report, are just generalizations using the usual proofs for scalar valued functions. On the other hand, some differences appear, as well. There is no Monotone Convergence theorem or Fatou's lemma, and the proof of the Dominated Convergence theorem is basically different.

MEASURABLE FUNCTIONS

Definition 1: A triple $(S,\Im,m)$, is called a measure space if:

1.1) S-set, $\Im-\sigma$ algebra:

(1.1.1) $\varnothing, S\in\Im$, (1.1.2) $A\in\Im \Rightarrow A^c\in\Im$, (1.1.3) $A_i\in\Im$ $\bigcup_{i=1}^{\infty}A_i\in\Im$).

1.2) $m:\Im\xrightarrow{measure}[0,\infty]$: (2.1) $m(\varnothing)=0$,

(1.2.1) $m(\varnothing)=0$, (1.2.2) $A\subseteq B,\ A,B\in\Im \Rightarrow m(A)\le m(B)$,

(1.2.3) If $\{A_i\}_{i=1}^{\infty}\subseteq\Im$ is a disjoint collection, then $m\left(\bigcup_{i=1}^{\infty}A_i\right)=\sum_{i=1}^{\infty}m(A_i)$.

Definition 2: $(S,\Im,m)$ is called σ-finite if $\exists\ \{B_i\}_{i=1}^{\infty}\in\Im$ such that $B_i\underset{i\to\infty}{\uparrow}S$ and $m(B_i)<\infty\quad\forall i\in\mathrm{N}$. Through this paper we always assume, that $(S,\Im,m)$ is at least σ-finite, if not finite.

Definition 3: A function $x_n(\bullet):S\longrightarrow X$ is called simple, if $x_n(s)=\sum_{i=1}^{m}c_i\chi_{E_i}(s)$, where $E_i\in\Im,\ \forall\ i$, and $x_n(\bullet)$ is zero off a set of finite measure.

Definition 4: A vector function $x(\bullet):S\to(X,\|\ \|_X)$ is said to be:

4.1) Strongly measurable if there exists a sequence of simple functions $\{x_n(\bullet)\}_{n=1}^{\infty}$ such that $x_n(s)\xrightarrow[n\to\infty]{\text{pointwise}}x(s),\ \forall s\in S$;

4.2) Weakly measurable if $\forall f\in X'\ f(x(\bullet))$ is a measurable scalar valued function.

Theorem 1: Let $(X,\|\ \|_X)$ be a separable Banach space. Then $x:S\to(X,\|\ \|_X)$ is strongly measurable if and only if $x^{-1}(U)$ is measurable for all U open in X (i.e. $x^{-1}(U)\in\Im$).

Proof: 1) Sufficiency: Suppose $x^{-1}(U)$ is measurable, for all U open in X. Then $x^{-1}(U)$ is measurable for any Borel set U. Since $x^{-1}(U)\in\Im$ implies $\left(x^{-1}(U)\right)^c=x^{-1}(U^c)\in\Im$. But $x^{-1}\left(\bigcup_{i=1}^{\infty}U_i\right)=\bigcup_{i=1}^{\infty}x^{-1}(U_i)$, and so the set $\Psi=\{U\subseteq X:\ x^{-1}(U)\in\Im\}$ forms a σ-algebra containing the open sets. Therefore, Ψ contains the Borel sets. Thus $x^{-1}(U)$ is measurable for any U-Borel.

Also, since X is separable $\exists \{a_n\}_{n=1}^{\infty}$, a dense subset of X.

Let $U_k^n = \left\{ z \in X: \|z - a_k\|_X \le \min\left\{\|z - a_m\|_X \;:\; 1 \le m \le n,\; m \ne k\right\}\right\}$.

Thus, $B_k^n = x^{-1}(U_k^n)$ is measurable. Also, let $\tilde{B}_k^n = B_k^n - \left(\bigcup_{i=1}^{k-1} B_i^n\right)$.

define: $x_n(s) = \sum_{i=1}^{n} a_k \chi_{\tilde{B}_k^n}(s), \quad n \in N$ (the closest approximation to $x(s)$ from $\{a_i\}_{i=1}^{n}$. Therefore, $x_n(s) \xrightarrow[n\to\infty]{} x(s)$, because $\{a_i\}_{i=1}^{\infty}$ is dense in X. Now, since $(S, \Im, m)$ σ-finite, $\exists \{B_i\}_{i=1}^{\infty} \in \Im$ such that $B_i \underset{i\to\infty}{\uparrow} S$ and $m(B_i) < \infty \quad \forall i \in \mathrm{N}$.

define: $y_n = \chi_{B_n} x_n$, $y_n(s) \xrightarrow[n\to\infty]{\text{pointwise}} x(s)$ in X, because for any $s \in X$ $y_n(s) = x_n(s)$ for large n. Also, clearly $y_n(\bullet)$ is a simple function, because it is 0 off a set of finite measure ($m(B_i) < \infty \quad \forall i \in \mathrm{N}$). Therefore, $x(\bullet)$ is strongly measurable. ⊗

2) Necessity: Let $x(\bullet)$ be strongly measurable thus $\exists \{x_n(\bullet)\}_{n=1}^{\infty}$ -simple such that $x_n(s) \xrightarrow[n\to\infty]{\text{pointwise}} x(s), \ \forall s \in S$. Let $\{a_i\}_{i=1}^{p}$ be the values of $x_n(s)$, so

$x_n(s) = \sum_{i=1}^{p} a_i \chi_{E_i}(s) \Rightarrow x_n^{-1}(W) = \{s:\ x_n(s) \in W\} = \bigcup_{a_i \in W} E_i$, where W is open in X, so $x_n^{-1}(W)$ is measurable. By same argument as in 1) $x_n^{-1}(W)$ is measurable for any Borel set W. Let U be any open set in X and let $\{V_n\}_{n=1}^{\infty}$ be a sequence of open sets satisfying $\overline{V}_n \subseteq U$, $\overline{V}_n \subseteq V_{n+1}$ $n \in N$ and $U = \bigcup_{n=1}^{\infty} V_n$, then $x^{-1}(V_p) \subseteq \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} x_k^{-1}(V_p) \subseteq x^{-1}(\overline{V}_p)$ and

$x^{-1}(U) = \bigcup_{p=1}^{\infty} x^{-1}(V_p) \subseteq \bigcup_{p=1}^{\infty}\left(\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} x_k^{-1}(V_p)\right) \subseteq \bigcup_{p=1}^{\infty} x^{-1}(\overline{V}_p) \subseteq x^{-1}(U)$, therefore

$x^{-1}(U) = \bigcup_{p=1}^{\infty}\left(\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} x_k^{-1}(V_p)\right)$ countable union of measurable sets, so $x^{-1}(U)$ is measurable for any open set U in X.

⊗

<u>Theorem 2:</u> (Pettis 1938) Let $(X,\| \|_X)$ be a separable Banach space and $x(\bullet):(S,\Im,m)\xrightarrow[measurable]{weakly}(X,\| \|_X)$, then $x(\bullet)$ is strongly measurable.

<u>Lemma 1:</u> If $(X,\| \|_X)$ be a separable Banach space and $B'=\{f\in X':\ \|f\|_{X'}\le 1\}$ be the unit ball in X', then there exists a sequence $\{f_n\}_{n=1}^{\infty}\subset B'$ such that for all $f_\circ\in B'$, there exists a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ of $\{f_n\}_{n=1}^{\infty}$ such that $f_\circ(x)=\lim_{k\to\infty} f_{n_k}(x)$ for all $x\in X$.

<u>Proof:</u> Let $\{a_i\}_{i=1}^{\infty}$ be a countable dense set in X. Consider the mapping $\varphi_n : B'\longrightarrow C^n$ given by $\varphi_n(f)=(f(a_1),\ f(a_2),\dots,\ f(a_n))$. C^n is separable and so is $\varphi_n(B')$ [see <u>Claim</u> on p. 6]. That is, there exists $\{f_k^n\}_{k=1}^{\infty}$ in B' so that $\{\varphi_n(f_k^n)\}_{k=1}^{\infty}$ is dense in $\varphi_n(B')$. Define $\{f_p\}_{p=1}^{\infty}\equiv\{f_k^n\}_{k,n=1}^{\infty}$, let $f_\circ\in B'$, choose $f_{p_l}\in\{f_p\}_{p=1}^{\infty}$ such that $|f_\circ(a_i)-f_{p_l}(a_i)|\le\left(\frac{1}{2}\right)^l$, $i=1,\ 2,\dots,\ l$. Therefore, $f_{p_l}(a_i)\xrightarrow[l\to\infty]{} f_\circ(a_i)$ for all a_i and this implies that $f_{p_l}(x)\xrightarrow[l\to\infty]{} f_\circ(x)$ for all $x\in X$ because $\{a_i\}_{i=1}^{\infty}$ is dense. ⊗

<u>Proof:</u> [Pettis theorem] (1) First we show that $x(\bullet)$ weakly measurable implies $\|x(\bullet)\|_X : S\longrightarrow[0,\infty]$ - measurable scalar valued function. Suppose $x(\bullet)$ weakly measurable, let $A=\{s\in S:\ \|x(s)\|_X\le a\}$, $A_f=\{s\in S:\ |f(x(s))|\le a\}$, observe that $A\subseteq \bigcap_{\|f\|_{X'}\le 1} A_f$. By Hahn-Banach theorem there exists $f_s(\bullet)$, such that $\|f_s\|_{X'}=1$ and $|f_s(x(s))|=\|x(s)\|_X$. Therefore $\bigcap_{\|f\|_{X'}\le 1} A_f\subseteq\bigcap_{f_s} A_{f_s}\subseteq A$, and so $A=\bigcap_{\|f\|_{X'}\le 1} A_f$. By lemma 1 there exists a sequence $\{f_p\}_{p=1}^{\infty}$ such that $\bigcap_{\|f\|\le 1} A_f=\bigcap_{p=1}^{\infty} A_{f_p}$. Thus A is measurable as a countable intersection of measurable sets (notice that $x(\bullet)$ weakly measurable implies $f(x(\bullet))$ measurable scalar valued function and so A_{f_p} are measurable sets).

(2) Now we observe that if $x(\bullet)$ is weakly measurable, then so is $[x(\bullet)-a]$, $a\in X$. Therefore, $x^{-1}(B(a,r))=\{s\in S:\ \|x(s)-a\|_X\le r\}$ is measurable for every ball $B(a,r)$. X is separable, thus X has a countable basis of such balls. So, if U is open in X then there exist balls $B(a,r)$ such that $U=\bigcup B(a,r)$ and $x^{-1}(U)=\bigcup x^{-1}(B(a,r))$ countable union of measurable sets, that means $x^{-1}(U)$ - measurable for all U open. By theorem 1, $x(\bullet)$ is strongly measurable. ⊗

Corollary 1: $x(\bullet)$ is strongly measurable if and only if $x(\bullet)$ is separably valued and weakly measurable.

Proof: 1) Necessity: Let $x(\bullet)$ be strongly measurable, then there exists a sequence of simple functions $x_n(s)\xrightarrow[n\to\infty]{\text{pointwise}}x(s)$, if $f\in X'$, then $f(x_n(s))\xrightarrow[n\to\infty]{\text{pointwise}}f(x(s))$ and $\{f(x_n(s)\}_{n=1}^{\infty}$ is a sequence of measurable functions (composition of continuous and measurable is a measurable function). Thus $f(x(\bullet))$ is measurable as a limit of measurable functions. To see that $x(\bullet)$ is separably valued we observe that $x_n(s)\xrightarrow[n\to\infty]{\text{pointwise}}x(s)$, where $x_n(s)$ are simple thus if $D=\{x_n(s):\ s\in S\}$, $\overline{D}$ is separable and contains the values of $x_n(s)$, $n\in N$, therefore $D_x=\{x(s):\ s\in S\}\subseteq\overline{D}$ and thus the following claim gives that D_x is separable;

Claim: Let M be a separable metric space and $T\subseteq M$. Then T is separable.

Proof: If M is separable and $\{a_n\}_{n=1}^{\infty}$ is dense in M, then

$\mathrm{B}=\left\{\{B(a_n,r)\}_{n=1}^{\infty}:\ r\in Q\cap(0,\infty)\right\}$ will be a countable basis of M.

Define: $\tilde{\mathrm{B}}=\{U\in\mathrm{B}:\ U\cap T\neq\varnothing\}$, since B is countable, $\tilde{\mathrm{B}}=\{U_j\}_{j=1}^{\infty}$.

Take $p_j\in U_j\cap T$, $\forall j\in N$, Therefore, $\{p_j\}_{j=1}^{\infty}$ will be dense in T.

⊗

2) Sufficiency: Let $x(\bullet)$ be weakly measurable and separably valued (i.e. $x(S)$ is separable). Let $\{y_p\}_{p=1}^{\infty}$ be dense in $x(S)$ and

$$Z=\left\{\sum_{k=1}^{n}(a_k+ib_k)y_k :\ n<\infty,\ a_k,b_k\in Q,\ y_k\in\{y_p\}_{p=1}^{\infty}\right\}$$

then Z is countable. Letting $Y=\overline{Z}$ implies that Y is separable, as a closure of a countable set, moreover Y is a Banach space with respect to the norm on X.

Now pick any $f\in Y'$, Y is a subspace of X, hence by Hahn-Banach theorem we can extend $f\in Y'$ to an element $\hat{f}$ of X' such that $\hat{f}/_Y=f$.

Recall that $x(\bullet)$ is weakly measurable, thus $\hat{f}(x(\bullet))$ is a measurable scalar valued function and $\hat{f}(x(s))=f(x(s)),\ \forall s\in S$. Applying Pettis theorem for $x(\bullet):S\xrightarrow[\text{measurable}]{\text{weakly}}Y$, a separable Banach space, we obtain that $x(\bullet)$ is strongly measurable.

⊗

<u>Theorem 3</u>: (Another version of Pettis theorem)

Let X' be separable and $y^*(\bullet):S\xrightarrow[\text{measurable}]{\text{weak}-*}X'$

$$\begin{array}{ccc} & & X'\ni y^* \\ & & \uparrow \\ s\in S & \longrightarrow & X\ni x \end{array}$$

(i.e. $\forall\ x\in X,\ y^*(\bullet)x$ is a measurable scalar valued function). Then $y^*(\bullet)$ is strongly measurable.

<u>Lemma 2:</u> Let $B=\{x\in X:\ \|x\|_X\le 1\}$ be the unit ball in X. Then if X' is separable, there exists a sequence $\{x_m\}_{m=1}^{\infty}\subset B$ satisfying:

For all $x\in B$ there exists a subsequence $\{x_{m_k}\}_{k=1}^{\infty}$ of $\{x_m\}_{m=1}^{\infty}$ such that $x^*(x_{m_k})\xrightarrow[k\to\infty]{}x^*(x)$, for all $x^*\in X'$.

Proof: (Lemma 2)

X' separable, let $D=\{x_k^*\}_{k=1}^{\infty}$ be dense in X'. As in the proof of Lemma 1, we define:

$\varphi_n : B \longrightarrow C^n$ by $\varphi_n(x)=\left(x_1^*(x),\ x_2^*(x),\ \dots,\ x_n^*(x)\right)$.

C^n is separable so there exists $\{x_k^n\}_{k=1}^{\infty} \subset B$, for all n, satisfying:

$$\{\varphi_n(x_k^n)\}_{k=1}^{\infty} \text{ dense in } \varphi_n(B).$$

Let $\{x_p\}_{p=1}^{\infty} \equiv \{x_k^n\}_{k,n=1}^{\infty}$, thus if $x \in B$ there exists a sequence $\{x_{p_q}\}_{q=1}^{\infty} \subset \{x_p\}_{p=1}^{\infty}$ satisfying $x^*(x_{p_q}) \xrightarrow[q\to\infty]{} x^*(x)$, for every $x^* \in D$.

Since D is dense in X', it follows that this holds for all $x^* \in X'$.

⊗

Proof: [Theorem 3]

(1) First we show that if $y^*(\bullet)$ is weak-* measurable then $\|y^*(\bullet)\|_{X'} : S \longrightarrow [0,\infty]$ is measurable.

Let $A=\{s \in S:\ \|y^*(s)\|_{X'} \le a\}$ and $A_x=\{s \in S:\ |y^*(s)(x)| \le a\}$.

Clearly, $A \subseteq \bigcap_{x\in B} A_x$, where B is the unit ball in X but if $|y^*(s)(x)| \le a,\ \forall x \in B$, then $\|y^*(s)\|_{X'} \le a$ and so $A \equiv \bigcap_{x\in B} A_x$.

Now by lemma 2, there exists a sequence $\{x_p\}_{p=1}^{\infty}$ in B such that $\bigcap_{x\in B} A_x \equiv \bigcap_{p=1}^{\infty} A_{x_p}$, therefore, $A \equiv \bigcap_{p=1}^{\infty} A_{x_p}$.

Note, that A_{x_p} is measurable for all p, since the mapping $s \longrightarrow y^*(s)(x_p)$ is measurable (remember, that $y^*(\bullet)$ is weak-* measurable by the hypothesis in theorem 3). Hence, A is measurable as a countable intersection of measurable sets.

(2) Since A is measurable, $\|y^*(\bullet)\|_{X'}$ is measurable. Let $a^* \in X'$, $y^*(\bullet)$ weak-* measurable, implies that so is $(y^*(\bullet)-a^*)$. Therefore, if $B(a^*,r)$ is any ball in X', $y^{*^{-1}}\left(B(a^*,r)\right)=\left\{s \in S:\ \|y^*-a^*\|_{X'}<r\right\}$ is measurable. But X' is separable, thus it has a countable basis of such balls (i.e. if U is open in X', then $U=\bigcup_{k<\infty} B_k(a^*,r)$ where $a^* \in X'$, $r>0$). Then $y^{*^{-1}}(U)=\bigcup_{k<\infty} y^{*^{-1}}\left(B_k(a^*,r)\right)$ is measurable as a countable union of measurable sets. Since X' is assumed to be separable, by theorem 1, $y^*(\bullet)$ is strongly measurable.

⊗

Theorem 4: If X' is separable, so is X.

Proof: Let $B=\left\{x \in X:\ \|x\|_X \le 1\right\}$ be the unit ball in X. Since X' is separable applying lemma 2, we obtain a sequence $\{x_m\}_{m=1}^{\infty} \subset B$ with the property: $\forall x \in B,\ \exists\left\{x_{m_p}\right\}_{p=1}^{\infty} \subset \{x_m\}_{m=1}^{\infty}$ such that whenever $x^* \in X'$, $x^*(x_{m_p}) \xrightarrow[p\to\infty]{} x^*(x)$. Let

$$V=\left\{\sum_{k=1}^{n}(a_k+ib_k)x_{m_k}\ :\ n<\infty,\ a_k,b_k \in Q,\ x_{m_k} \in \{x_m\}_{m=1}^{\infty}\right\},$$

then $\overline{V}$ is a separable subspace of X. We claim that: $\overline{V} \equiv X$.

If not, there exists an element $x_\circ \in X-\overline{V}$. By the separation theorem, there exists $x_\circ^* \in X'$ satisfying $x_\circ^*(x_\circ) \ne 0,\ x_\circ^*(v)=0\ \forall v \in \overline{V}$. Definitely, $x_\circ \ne 0$, and $\frac{x_\circ}{\|x_\circ\|_X} \in B$.

Hence, $x_\circ^*\left(\frac{x_\circ}{\|x_\circ\|_X}\right)=\lim_{k\to\infty}\left(x_\circ^*(x_{m_k})\right)$, for some $\left\{x_{m_k}\right\}_{k=1}^{\infty} \subset \{x_m\}_{m=1}^{\infty}$.

But $x_\circ^*(x_{m_k})=0,\ \forall m_k$, thus $x_\circ^*\left(\frac{x_\circ}{\|x_\circ\|_X}\right)=0$, which contradicts $x_\circ^*(x_\circ) \ne 0$. Therefore, $\overline{V} \equiv X$ and so X is separable.

⊗

Corollary 2: If X is reflexive Banach space, then X is separable if and only if X' is separable.

Proof: 1) Sufficiency is given by theorem 4, X' separable $\Rightarrow X$ separable;

2) Necessity: Let X be separable and reflexive. Then the mapping $\theta : X \xrightarrow[\text{onto}]{1-1} X''$, defined by $\theta(x)x^* = \overline{x^*(x)}$ will be onto.

$$
\begin{array}{ccc}
 & \therefore \quad X'' \ni (\theta x)x^* & \\
 & \uparrow & \\
x \in X \longrightarrow & X' & \ni x^*
\end{array}
$$

In addition, θ is linear and continuous, so if $\{x_n\}_{n=1}^{\infty}$ is dense in X, then $\{(\theta x_n)\}_{n=1}^{\infty}$ will be a countable dense set in the double-dual space X''. Now we apply theorem 4, for X' and X'', to get that the separability of X'' implies separability of X'.

$\otimes$

THE BOCHNER INTEGRAL

Definition 5: Let $x(\bullet):(S,\mathfrak{S},m)\xrightarrow[\text{measurable}]{\text{strongly}}(X,\|\ \|_X)$, where $(S,\mathfrak{S},m)$ is a finite measure space and $(X,\|\ \|_X)$ is a Banach space. Then $x(\bullet)$ is Bochner integrable if:

1) There exists a sequence of simple functions $\{x_n\}_{n=1}^{\infty}$, such that $x_n(s)\xrightarrow[n\to\infty]{\text{pointwise}}x(s),\ s\in S$;

2) And $\int_S \|x_n(s)-x_k(s)\|_X dm \xrightarrow[n,k\to\infty]{} 0$.

Definition 6: 6.1) If $x_n(s)=\sum_{k=1}^{p} c_k\chi_{E_k}(s)$ is simple then $\int_S x_n(s)dm=\sum_{k=1}^{p} c_k m(E_k)$;

6.2) If $x(\bullet)$ is Bochner integrable, then $\int_S x(s)dm=\lim_{k\to\infty}\int_S x_k(s)dm$.

Remark: Note that $(X,\|\ \|_X)$ will be in general an infinite diminsional space, thus c_k are no more constants, rather they are infinite dimensional vectors. Now we will be showing that the Bochner integral is well defined.

Proposition 1: The Bochner Integral is well defined on simple functions.

Proof: Suppose $x_n(s)=\sum_{k=1}^{p} c_k\chi_{E_k}(s)=\sum_{l=1}^{q} d_l\chi_{F_l}(s),\ \forall s\in S$. We need to show that $\sum_{k=1}^{p} c_k m(E_k)=\sum_{l=1}^{q} d_l m(F_l)$. Let $\Lambda\in X'$, then $\Lambda\left(\sum_{k=1}^{p} c_k\chi_{E_k}(s)\right)=\Lambda\left(\sum_{l=1}^{q} d_l\chi_{F_l}(s)\right),\ \forall s\in S$. Consequently, since the integral is well defined on scalar valued functions in $L^1(S,C)$, we obtain $\int_S \Lambda\sum_{k=1}^{p} c_k\chi_{E_k}(s)dm=\int_S \Lambda\sum_{l=1}^{q} d_l\chi_{F_l}(s)dm$. Also, note that: $\int_S \Lambda\sum_{k=1}^{p} c_k\chi_{E_k}(s)dm=\int_S\sum_{k=1}^{p}\Lambda(c_k)\chi_{E_k}(s)dm=\sum_{k=1}^{p}\Lambda(c_k)m(E_k)=\Lambda\left(\sum_{k=1}^{p} c_k m(E_k)\right)$

$$\int_S \Lambda\sum_{l=1}^{q} d_l\chi_{F_l}(s)dm = \int_S \sum_{l=1}^{q}\Lambda(d_l)\chi_{F_l}(s)dm = \sum_{l=1}^{q}\Lambda(d_l)m(F_l) = \Lambda\left(\sum_{l=1}^{q} d_l m(F_l)\right).$$

Hence, $\Lambda\left(\sum_{k=1}^{p} c_k m(E_k)\right) = \Lambda\left(\sum_{l=1}^{q} d_l m(F_l)\right)$. Now since $\Lambda \in X'$ was arbitrary we have that: $\sum_{k=1}^{p} c_k m(E_k) = \sum_{l=1}^{q} d_l m(F_l)$.

⊗

Remarks: (1) The Bochner integral is linear on simple functions:

$$\int_S ax_n(s)dm = \int_S \sum_{k=1}^{p} ac_k\chi_{E_k}(s)dm = \sum_{k=1}^{p} ac_k m(E_k) =$$

$$a\sum_{k=1}^{p} c_k m(E_k) = a\int_S \sum_{k=1}^{p} c_k\chi_{E_k}(s)dm = a\int_S x_n(s)dm;$$

Let $x_n(s) = \sum_{k=1}^{p} c_k\chi_{E_k}(s)$, $y_m(s) = \sum_{l=1}^{q} b_l\chi_{F_l}(s)$, then $x_n(s) + y_m(s) = \sum_{k=1}^{p} c_k\chi_{E_k}(s) + \sum_{l=1}^{q} d_l\chi_{F_l}(s)$

will be a simple function and thus by the very definition $\int_S x_n(s) + y_m(s)dm = \sum_{k=1}^{p} c_k m(E_k) + \sum_{l=1}^{q} d_l m(F_l)$. Here we used the fact that the Bochner integral is well defined on simple functions.

⊗

(2) $\int_S \|x_n(s)\|_X dm \geq \left\|\int_S x_n(s)dm\right\|_X$ for simple functions. To show this let $x_n(s) = \sum_{k=1}^{p} c_k\chi_{E_k}(s)$ and $\{d_i\}_{i=1}^{m}$ be the set of all distinct non-zero values of $x_n(s)$. Also, let $F_i = x_n^{-1}(d_i)$, $i = 1,\ldots m$, thus $x_n(s) = \sum_{i=1}^{m} d_i\chi_{F_i}(s)$.

Observe, that $\|x_n(s)\|_X = \sum_{i=1}^{m}\|d_i\|_X\chi_{F_i}(s)$, since $\forall s \in S \Rightarrow s \in F_{i_0}$ for some i_0 and $s \notin \bigcup_{i \neq i_0} F_i$, therefore, $\|x_n(s)\|_X = \|d_{i_0}\|_X = \sum_{i=1}^{m}\|d_i\|_X\chi_{F_i}(s)$.

Hence,

$$\int_S \|x_n(s)\|_X dm = \int_S \sum_{i=1}^{m} \|d_i\|_X \chi_{F_i}(s) = \sum_{i=1}^{m} \|d_i\|_X m(F_i) \geq \left\| \sum_{i=1}^{m} d_i m(F_i) \right\|_X = \left\| \int_S x_n(s) dm \right\|_X ;$$

⊗

(3) If $\Lambda \in X'$, $x_n(s) = \sum_{k=1}^{p} c_k \chi_{E_k}(s)$, then $\int_S \Lambda \sum_{k=1}^{p} c_k \chi_{E_k}(s) dm = \Lambda \int_S \sum_{k=1}^{p} c_k \chi_{E_k}(s) dm$:

$$\int_S \Lambda \sum_{k=1}^{p} c_k \chi_{E_k}(s) dm = \sum_{k=1}^{p} \int_S \Lambda(c_k) \chi_{E_k}(s) dm = \sum_{k=1}^{p} \Lambda(c_k) m(E_k) =$$
$$= \Lambda\left(\sum_{k=1}^{p} c_k m(E_k) \right) = \Lambda \int_S \sum_{k=1}^{p} c_k \chi_{E_k}(s) dm .$$

⊗

Theorem 5: Let $x(\bullet)$ be Bochner integrable, then the Bochner integral $\int_S x(s) dm$ is well defined.

Proof: Suppose $\{x_n^i\}_{n=1}^{\infty}$, $i = 1,2$ are two sequences of simple functions such that

$$x_n^i(s) \xrightarrow[n\to\infty]{i=1,2} x(s), \ \forall s \in S, \ \int_S \left\| x_n^i(s) - x_k^i(s) \right\|_X dm \xrightarrow[k,n\to\infty]{} 0, \ i = 1,2.$$

We need to show that $\int_S x_n^1(s) dm$ and $\int_S x_n^2(s) dm$ have the same limit, as $n \to \infty$ and that the limit exists.

1) Existance of the limits:

$$\left\| \int_S x_n^i(s) dm - \int_S x_k^i(s) dm \right\|_X = \left\| \int_S (x_n^i(s) - x_k^i(s)) dm \right\|_X \overset{(2)}{\leq} \int_S \left\| x_n^i(s) - x_k^i(s) \right\|_X dm, \ i = 1,2,$$

which is assumed to converge to 0. Thus, $\left\{ \int_S x_n^i(s) dm \right\}_{n=1}^{\infty}$ forms a Cauchy sequence in a Banach space. Therefore, $\exists \lim_{n\to\infty} \int_S x_n^i(s) dm, \ i = 1,2;$

2) $$\left\|\int_S x_n^1(s)dm - \int_S x_k^2(s)dm\right\|_X \le \left\|\int_S \left(x_n^1(s) - x_k^2(s)\right)dm\right\|_X \le \int_S \left\|x_n^1(s) - x_k^2(s)\right\|_X dm \le$$

$$\le \int_S \left\|x_n^1(s) - x(s)\right\|_X dm + \int_S \left\|x(s) - x_k^2(s)\right\|_X dm \le$$

$$\left\{\begin{array}{c}\text{Since: } \left\|x_n^i(s) - x(s)\right\|_X \ge 0,\ \left\|x_n^i(s) - x(s)\right\| = \lim_{k\to\infty}\left\|x_n^i(s) - x_k^i(s)\right\|_X,\ i = 1,2 \\ \text{Applying Fatou's lemma we obtain:}\end{array}\right\}$$

$\le \liminf_{k\to\infty} \int_S \left\|x_n^1(s) - x_k^1(s)\right\|_X dm + \liminf_{n\to\infty} \int_S \left\|x_n^2(s) - x_k^2(s)\right\|_X dm < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$, since $\int_S \left\|x_n^i(s) - x_m^i(s)\right\|_X dm < \frac{\varepsilon}{2}$, $i = 1,2$ whenever n and m are large enough.

⊗

Theorem 6: Let $x(\bullet)$ be defined on a finite measure space $(S, \mathfrak{S}, m)$, then $x(\bullet)$ is Bochner integrable if and only if $x(\bullet)$ is strongly measurable and $\int_S \|x(s)\|_X dm < \infty$.

Proof: 1) Suppose that $x(\bullet)$ is Bochner integrable, then $x(\bullet)$ is strongly measurable and there exists a sequence of simple functions $\{x_n(\bullet)\}_{n=1}^{\infty}$, $x_n(s) \xrightarrow[n\to\infty]{\text{pointwise}} x(s)$. By Fatou's lemma

$$\left|\int_S \|x_n(s)\|_X dm - \int_S \|x(s)\|_X dm\right| \le \int_S \|x_n(s) - x(s)\|_X dm \le \liminf_{m\to\infty} \int_S \|x_n(s) - x_m(s)\|_X dm$$

thus $\int_S \|x(s)\|_X dm \le \int_S \|x_n(s)\|_X dm + \liminf_{m\to\infty} \int_S \|x_n(s) - x_m(s)\|_X dm$, now if n and m are large $\int_S \|x_n(s) - x_m(s)\|_X dm < 1$, therefore, $\int_S \|x(s)\|_X dm \le \int_S \|x_n(s)\|_X dm + 1 < \infty$.

2) To show the other direction, we let $x(\bullet)$ be strongly measurable and $\int_S \|x(s)\|_X dm < \infty$. Define $G_n = \left\{s \in S\colon \|x_n(s)\|_X \le 2\|x(s)\|_X\right\}$ and

$$y_n(s) = x_n(s)\chi_{G_n}(s) = \begin{cases} x_n(s), & s \in G_n \ (i.e. \|x_n(s)\|_X \le 2\|x(s)\|_X) \\ 0, & s \notin G_n \ (i.e.\ \|x_n(s)\|_X > 2\|x(s)\|_X) \end{cases}.$$

Therefore, if $x(s)=0$ then $y_n(s)=0,\ \forall n\in N$ and if $x(s)\neq 0$ then

$$y_n(s)\xrightarrow[n\to\infty]{\text{pointwise}}x(s)$$

{since $x_n(s)\xrightarrow[n\to\infty]{\text{pointwise}}x(s) \Rightarrow \|x_n(s)\|_X\xrightarrow[n\to\infty]{\text{pointwise}}\|x(s)\|_X$ }.

Furthermore, $\|y_n(s)\|_X\le 2\|x(s)\|_X,\ \forall n\in N,\ s\in S$, then $\|y_n(s)-y_m(s)\|_X\le\|y_n(s)\|_X+\|y_m(s)\|_X\le 4\|x(s)\|_X\in L^1(S)$. Remember, that $\|y_n(s)-y_m(s)\|_X\xrightarrow[m,n\to\infty]{}0$, so applying the Dominated Convergence Theorem (for $L^1(S)$ functions) we obtain: $\lim\limits_{m,n\to\infty}\int\limits_S\|y_n(s)-y_m(s)\|_X dm=\int\limits_S\lim\limits_{m,n\to\infty}\|y_n(s)-y_m(s)\|_X dm=0$ and hence $\int\limits_S\|y_n(s)-y_m(s)\|_X dm\xrightarrow[m,n\to\infty]{}0$. Then $\{y_n(\bullet)\}_{n=1}^{\infty}$ is the desired sequence of simple functions, that makes $x(\bullet)$ Bochner integrable.

⊗

Proposition 2: Let $x(\bullet)$ be Bochner integrable, $\Lambda\in X'$. Then

$$1)\ \left\|\int\limits_S x(s)dm\right\|_X\le\int\limits_S\|x(s)\|_X dm;\quad 2)\ \int\limits_S\Lambda(x(s))dm=\Lambda\left(\int\limits_S x(s)dm\right).$$

Proof: 1) Suppose $x(\bullet)$ is Bochner integrable, then $\exists\{x_n(\bullet)\}_{n=1}^{\infty}$ a sequence of simple functions such that $x_n(s)\xrightarrow[n\to\infty]{\text{pointwise}}x(s)$, as above we let

$$y_n(s)=x_n(s)\chi_{G_n}(s)=\begin{cases}x_n(s), & s\in G_n\ (i.e.\|x_n(s)\|_X\le 2\|x(s)\|_X)\\ 0, & s\notin G_n\ (i.e.\ \|x_n(s)\|_X>2\|x(s)\|_X)\end{cases},\ \text{thus}\ y_n(s)\xrightarrow[n\to\infty]{\text{pointwise}}x(s)$$

since $x_n(s)\xrightarrow[n\to\infty]{\text{pointwise}}x(s)$. By the Dominated Convergence Theorem

$$\lim_{n\to\infty}\int\limits_S\|y_n(s)-x(s)\|_X dm=\int\limits_S\lim_{n\to\infty}\|y_n(s)-x(s)\|_X dm=0,\ \lim_{n\to\infty}\int\limits_S\|y_n(s)\|_X dm=\int\limits_S\|x(s)\|_X dm$$

and $\lim\limits_{n,m\to\infty}\int\limits_S\|y_n(s)-y_m(s)\|_X dm=0$.

Then by definition $\int\limits_S x(s)dm=\lim\limits_{n\to\infty}\int\limits_S y_n(s)dm$. Since the norm $\|\ \|_X$ is a continuous function $\left\|\int\limits_S x(s)dm\right\|_X=\lim\limits_{n\to\infty}\left\|\int\limits_S y_n(s)dm\right\|_X\le\lim\limits_{n\to\infty}\int\limits_S\|y_n(s)\|_X dm=\int\limits_S\|x(s)\|_X dm$.

2) Let $\Lambda \in X'$, then

$$\Lambda\left(\int_S x(s)dm\right) = \Lambda\left(\lim_{n\to\infty}\int_S y_n(s)dm\right) = \lim_{n\to\infty}\Lambda\left(\int_S y_n(s)dm\right) = \lim_{n\to\infty}\int_S \Lambda(y_n(s))dm \overset{*}{=}$$

$$\overset{*}{=} \int_S \lim_{n\to\infty}\Lambda(y_n(s))dm = \int_S \Lambda(x(s))dm.$$ Equality * is justified by the Dominated Convergence Theorem, since

$$|\Lambda y_n(s)| \le \|\Lambda\|_{X'}\|y_n(s)\|_X \le 2\|\Lambda\|_{X'}\|x(s)\|_X \in L^1(S,X)$$ ⊗

Definition 7: $$L^p(S,X) = \left\{ x(\bullet) : S \xrightarrow[\text{integrable}]{\text{Bochner}} X :\ \|x(\bullet)\|_p = \left[\int_S \|x(s)\|_X^p dm\right]^{\frac{1}{p}} < \infty \right\},\ p \ge 1.$$

Lemma 3: Let $\{x_n(\bullet)\}_{n=1}^{\infty}$ be a Cauchy sequence in $L^p(S,X)$ such that $\sum_{n=1}^{\infty}\|x_{n+1}(\bullet) - x_n(\bullet)\|_p < \infty$. Then there exists $x(\bullet) \in L^p(S,X)$ satisfying:

$$x_n(s) \xrightarrow[n\to\infty]{\text{a.e.}} x(s),\ \|x(\bullet) - x_n(\bullet)\|_p \xrightarrow[n\to\infty]{} 0.$$

Proof: 1) Define: $g_N(s) = \sum_{n=1}^{N}\|x_{n+1}(s) - x_n(s)\|_X$, thus $\|g_N(\bullet)\|_p = \left[\int_S \|g_N(s)\|_X^p dm\right]^{\frac{1}{p}} \le$

by Minkowski's inequality for real valued functions

$$\le \sum_{n=1}^{N}\left(\int_S \|x_{n+1}(s) - x_n(s)\|_X^p dm\right)^{\frac{1}{p}} = \sum_{n=1}^{N}\|x_{n+1}(\bullet) - x_n(\bullet)\|_p \le \sum_{n=1}^{\infty}\|x_{n+1}(\bullet) - x_n(\bullet)\|_p < \infty.$$

Hence $g_N(\bullet) \in L^p(S)$, N-natural. Letting $g(s) = \lim_{N\to\infty} g_N(s)$ implies that $g_N(s) \underset{N\to\infty}{\uparrow} g(s),\ \forall s \in S$. By the Monotone Convergence Theorem (for real valued functions), $g(\bullet) \in L^p(S)$ and

$$\lim_{N\to\infty}\int_S g_N(s)dm = \int_S g(s)dm.$$ Therefore, $$\|g(\bullet)\|_p = \left[\int_S \|g(s)\|_X^p dm\right]^{\frac{1}{p}} =$$

$$= \lim_{N\to\infty}\left[\int_S \|g_N(s)\|_X^p dm\right]^{\frac{1}{p}} = \lim_{N\to\infty}\|g_N(\bullet)\|_p \le \lim_{N\to\infty}\sum_{n=1}^{N}\|x_{n+1}(\bullet) - x_n(\bullet)\|_p = \sum_{n=1}^{\infty}\|x_{n+1}(\bullet) - x_n(\bullet)\|_p < \infty.$$

There exists a set E, such that $\forall s \in S-E,\ g(s) < \infty$ and $m(E)=0$.
Now we observe that $x_1(s) \in L^p(S,X)$ and

$$x_{N+1}(s) = \left(x_{N+1}(s) - x_1(s)\right) + x_1(s) = \sum_{n=1}^{N}\left(x_{n+1}(s) - x_n(s)\right) + x_1(s),$$

then $\sum_{n=1}^{\infty}\left(x_{n+1}(s) - x_n(s)\right)$ is absolutely summable for $s \in S-E$.
Furthermore, the space X is complete and therefore this sum is summable (i.e. $\sum_{n=1}^{\infty}\left(x_{n+1}(s) - x_n(s)\right)$ converges to an element in X).
In conclusion, $\forall s \in S-E,\ \exists \lim_{N\to\infty} x_N(s)$, we will call this limit

$$x(s) = \begin{cases} \lim_{N\to\infty} x_N(s), & s \in S-E \\ 0, & s \in E \end{cases}$$

. Since $x_N(\bullet) \in L^p(S,X)$, then $x_N(\bullet)$ is strongly measurable, by the necessity of corollary 1, $x_N(\bullet)$ is separably valued and weakly measurable. Hence $x(\bullet)$ is separably valued, as a limit of such functions. Also, if $f \in X'$ then $f(x_n(\bullet))$ will be measurable and $\lim_{n\to\infty}\left(f(x_n(s))\right) = f(x(s))$, for all $s \in S-E$ and $f(x(s)) = 0,\ s \in E$.
We obtain, that $f(x(\bullet))$ is measurable, as a limit of measurable functions. Thus $x(\bullet)$ is weakly measurable and therefore strongly measurable (by the sufficiency of corollary 1).

2) Now we apply Fatou's lemma (for $\Re$-valued functions) to get: $\int_S \|x(s) - x_N(s)\|_X^p\, dm \le \lim_{M\to\infty}\inf \int_S \|x_M(s) - x_N(s)\|_X^p\, dm$. Let $\varepsilon > 0$ be arbitrary. Since $\sum_{N=1}^{\infty}\|x_{N+1}(\bullet) - x_N(\bullet)\|_p < \infty$, applying the triangle inequality for norms:

$$\left(\int_S \|x_M(s) - x_N(s)\|_X^p\, dm\right)^{\frac{1}{p}} \le \sum_{i=N}^{M-1}\left(\int_S \|x_{i+1}(s) - x_i(s)\|_X^p\, dm\right)^{\frac{1}{p}} = \sum_{i=N}^{M-1}\|x_i(\bullet) - x_{i+1}(\bullet)\|_p < \varepsilon,$$

provided that $M > N$, and N is large enough.

Therefore, $\|x(\bullet) - x_N(\bullet)\|_p < \varepsilon$, whenever N is large.

We observe, that $x(\bullet)\in L^p(S,X)$, by Minkowski's inequality

$$\|x(\bullet)\|_p=\left[\int_S\|x(s)\|_X^p dm\right]^{\frac{1}{p}}\le\left[\int_S\|x_N(s)\|_X^p dm\right]^{\frac{1}{p}}+\left[\int_S\|x(s)-x_N(s)\|_X^p dm\right]^{\frac{1}{p}},\text{ and so}$$

$\|x(\bullet)\|_p\le\|x_N(\bullet)\|_p+\varepsilon<\infty$, for N large. This completes the proof of lemma 3.

⊗

Theorem 7: $L^p(S,X)$ is a Banach space. Moreover, any Cauchy sequence has a subsequence that converges almost everywhere (a.e.).

Proof: Let $\{x_n(\bullet)\}_{n=1}^{\infty}\in L^p(S,X)$ be a Cauchy sequence. We can extract a subsequence $\{x_{n_k}(\bullet)\}_{k=1}^{\infty}\subseteq\{x_n(\bullet)\}_{n=1}^{\infty}$ satisfying: $\left\|x_{n_k}(\bullet)-x_{n_{k+1}}(\bullet)\right\|_p\le\frac{1}{2^k}$, $\forall k\in N$.

Then, $\sum_{k=1}^{\infty}\left\|x_{n_k}(\bullet)-x_{n_{k+1}}(\bullet)\right\|_p\le\sum_{k=1}^{\infty}\frac{1}{2^k}=1<\infty$, lemma 3 gives the existance of an element $x(\bullet)\in L^p(S,X)$, such that, $x_{n_k}(s)\xrightarrow[k\to\infty]{\text{a.e}}x(s)$ and $\left\|x_{n_k}(\bullet)-x(\bullet)\right\|_p\xrightarrow[k\to\infty]{}0$. Therefore, if N,k are sufficiently large

$$\|x_N(\bullet)-x(\bullet)\|_p\le\left\|x_{n_k}(\bullet)-x(\bullet)\right\|_p+\left\|x_{n_k}(\bullet)-x_N(\bullet)\right\|_p<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon,\text{ since}$$

$\{x_n(\bullet)\}_{n=1}^{\infty}\in L^p(S,X)$ was chosen to be Cauchy.

⊗

Remark: Clearly, Fatou's lemma and Monotone Convergence theorem make no sense for vector valued functions. However, the Dominated Convergence theorem does and here is how it works.

Theorem 8: Let $x(\bullet)$ and $x_n(\bullet)$ be strongly measurable, and $x_n(s)\xrightarrow[n\to\infty]{\text{a.e.}}x(s)$. Also, let $g(s)\in L^1(S,R^+)$ satisfy $\|x_n(s)\|_X\overset{\text{a.e.}}{\le}g(s)$, $\forall n\in N$. Then:

1) $x(\bullet)$ is Bochner integrable;

2) $\int_S x(s)dm=\lim_{n\to\infty}\int_S x_n(s)dm$.

Proof: 1) Suppose $x_n(s) \xrightarrow[n\to\infty]{\text{a.e.}} x(s)$, then $\|x_n(s)-x(s)\|_X \le \|x_n(s)\|_X + \|x(s)\|_X \overset{\text{a.e.}}{\le} 2g(s)$.

Thus, by the usual Dominated Convergence theorem (for real valued functions) $0 = \int_S \lim_{n\to\infty} \|x_n(s)-x(s)\|_X dm = \lim_{n\to\infty} \int_S \|x_n(s)-x(s)\|_X dm$.

Also, if $\varepsilon > 0$ is arbitrary, letting m, n be large enough implies:

$$\|x_n(\bullet)-x_m(\bullet)\|_1 = \int_S \|x_n(s)-x_m(s)\|_X dm \le \int_S \|x_n(s)-x(s)\|_X dm + \int_S \|x(s)-x_m(s)\|_X dm < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon.$$

Therefore, $\{x_n(\bullet)\}_{n=1}^{\infty}$ is Cauchy in $L^1(S,X)$. By theorem 7, there exists an element $y(\bullet) \in L^1(S,X)$ and a subsequence

$$\{x_{n_k}(\bullet)\}_{k=1}^{\infty} \subset \{x_n(\bullet)\}_{n=1}^{\infty}, \text{ such that } x_{n_k}(s) \xrightarrow[k\to\infty]{\text{a.e.}} y(s).$$

However, $\lim_{k\to\infty} x_{n_k}(s) \overset{\text{a.e.}}{=} x(s)$, thus $y(s) \overset{\text{a.e.}}{=} x(s)$. And so, $\infty > \|y(\bullet)\|_1 = \|x(\bullet)\|_1$.

Now we apply theorem 6 to get that $x(\bullet)$ is Bochner integrable.

2) To show the second part we use the hypothesis and part one:

$$\left\| \int_S x(s)dm - \int_S x_n(s)dm \right\|_X \le \int_S \|x_n(s)-x(x)\|_X dm \xrightarrow[n\to\infty]{} 0 .$$

Therefore, $\int_S x(s)dm = \lim_{n\to\infty} \int_S x_n(s)dm$.

⊗

VECTOR MEASURES

Definition 8: Let $(S,\Im)$ be a pair of a set and a σ-algebra on it. Then the function $F:\Im \longrightarrow (Y,\|\ \|_Y)$-Banach space, is called a vector measure if: $F\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} F(E_i)$, for any disjoint union of sets in $\Im$.

Definition 9: Suppose $E \in \Im$, $\Pi(E) = \left\{ \{E_i\}_{i=1}^n - \text{disjoint}: \bigcup_{i=1}^{n} E_i = E,\ E_i \in \Im,\ i = 1,\dots,n \right\}$ is called partition of the set E.

Definition 10: Let F be a vector measure, then

$$|F|(E) = \sup\left\{ \sum_{i=1}^{n} \|F(E_i)\|_Y : \ \Pi(E) = \{E_i\}_{i=1}^n - \text{finite partition of } E \right\}$$

is called the total variation of F.

Proposition 3: Let $|F|(S) < \infty$, then $|F|$ is a measure (i.e. $(S,\Im,|F|)$ is a measure space).

Proof: 1) Suppose $E_1 \cap E_2 = \varnothing$ and $\varepsilon > 0$ is arbitrary. Since $|F|(S) < \infty$, then $|F|(E_k) < \infty$, $k = 1,2$. Thus, there exist partitions $\Pi(E_k) = \{E_k^i\}_{i=1}^{n_k}$, $k = 1,2$, such that $|F|(E_k) - \varepsilon < \sum_{i=1}^{n_k} \|F(E_k^i)\|_Y$, $k = 1,2$. Let $\Pi(E_1 \cup E_2) = \Pi(E_1) \cup \Pi(E_2)$, then

$$|F|(E_1 \cup E_2) \geq \sum_{E \in \Pi(E_1 \cup E_2)} \|F(E)\|_Y = \sum_{i=1}^{n_1} \|F(E_1^i)\|_Y + \sum_{i=1}^{n_2} \|F(E_2^i)\|_Y > |F|(E_1) - \varepsilon + |F|(E_2) - \varepsilon.$$

Because $\varepsilon > 0$ is arbitrary small, $|F|(E_1 \cup E_2) \geq |F|(E_1) + |F|(E_2)$.

By induction, $|F|\left(\bigcup_{i=1}^{n} E_i\right) \geq \sum_{i=1}^{n} |F|(E_i)$, $n < \infty$ and so

$$|F|\left(\bigcup_{i=1}^{\infty} E_i\right) \geq |F|\left(\bigcup_{i=1}^{n} E_i\right) \geq \sum_{i=1}^{n} |F|(E_i),\ n < \infty \ \Rightarrow |F|\left(\bigcup_{i=1}^{\infty} E_i\right) \geq \sum_{i=1}^{\infty} |F|(E_i).$$

2) Suppose $\{E_i\}_{i=1}^{\infty}$ is a disjoint sequence of sets in $\Im$. Denote $E_\infty = \bigcup_{i=1}^{\infty} E_i$, $|F|(E_\infty) < \infty$. Let $\varepsilon > 0$, there exists a partition $\Pi(E_\infty) = \{A_i\}_{i=1}^{n}$, so that $|F|(E_\infty) - \varepsilon < \sum_{i=1}^{n} \|F(A_i)\|_Y$. Observe, that $\{A_i \cap E_k\}_{k=1}^{\infty}$ is a disjoint collection. Now since F is a vector measure,

$$\|F(A_i)\|_Y = \left\| F\left(\bigcup_{k=1}^{\infty} (E_k \cap A_i)\right)\right\|_Y = \left\|\sum_{k=1}^{\infty} F(A_i \cap E_k)\right\|_Y \le \sum_{k=1}^{\infty} \|F(A_i \cap E_k)\|_Y, \quad i = 1,\dots,n.$$ Then

$$|F|(E_\infty) - \varepsilon < \sum_{i=1}^{n}\sum_{k=1}^{\infty} \|F(A_i \cap E_k)\|_Y \overset{(*)}{=} \sum_{k=1}^{\infty}\sum_{i=1}^{n} \|F(A_i \cap E_k)\|_Y \le \sum_{k=1}^{\infty} |F|(E_k),$$ because

$\bigcup_{i=1}^{n} (E_k \cap A_i) = E_k$, $\forall k$ and $|F|(E_k)$ is the supremum of such partitions.

Since $\varepsilon > 0$ is arbitrary small, we obtain $|F|(E_\infty) \le \sum_{k=1}^{\infty} |F|(E_k)$.

Finally, 1) and 2) imply that $|F|(E_\infty) = \sum_{k=1}^{\infty} |F|(E_k)$ and hence $|F| : S \longrightarrow [0,\infty]$ is a measure.

(*) Fubini's theorem for counting measure justifies this equality.

⊗

<u>Definition 11:</u> A Banach space $(Y, \|\ \|_Y)$ is said to have the Radon-Nikodym Property if: Assuming

1) $\exists\ F : \Im \xrightarrow[\text{measure}]{\text{vector}} (Y, \|\ \|_Y)$ with $|F|(S) < \infty$, where $(S, \Im, m)$ is a finite measure space and
2) $F(E) = 0$ whenever $m(E) = 0$ (i.e. $F << m$, F is absolutely continuous with respect to m).

It follows that: $\exists\ g(\bullet) \in L^1(S,Y)$ such that $F(E) = \int_E g(s)dm,\ \ \forall E \in \Im$.

<u>Remark:</u> Some Banach spaces have the Radon-Nikodym Property and some do not. Now we will identify some classes of spaces that do have it.

<u>Theorem 9:</u> Let X' be a separable anti-dual space. Then X' has the Radon-Nikodym Property.

Proof: Let $F:\Im \xrightarrow[\text{measure}]{\text{vector}} X'$, $(S,\Im,m)$ be finite measure space, $|F|(S)<\infty$ and $F \ll m$. Let $x \in X$ be fixed.

(1) Consider the mapping: $F_x:\Im \longrightarrow C$, such that $F_x(E)=F(E)x$. First we will show that F_x is a complex measure absolutely continuous with respect to the total variation of F (i.e. $F_x \ll |F|$).

(1.1) Let $\{E_i\}_{i=1}^{\infty}$ be a disjoint collection of sets in $\Im$. Then, since F is a vector measure, we have the following:

$$F_x\left(\bigcup_{i=1}^{\infty} E_i\right)=\left(F\left(\bigcup_{i=1}^{\infty} E_i\right)\right)(x)=\left(\sum_{i=1}^{\infty} F(E_i)\right)(x)=\left(\lim_{n\to\infty}\sum_{i=1}^{n} F(E_i)\right)(x)=$$

$$=\overline{\theta(x)\left(\lim_{n\to\infty}\sum_{i=1}^{n} F(E_i)\right)}=\overline{\lim_{n\to\infty}\left(\theta(x)\sum_{i=1}^{n} F(E_i)\right)}=\overline{\lim_{n\to\infty}\left(\sum_{i=1}^{n}\overline{F(E_i)(x)}\right)}=\sum_{i=1}^{\infty} F(E_i)(x)=\sum_{i=1}^{\infty} F_x(E_i).$$

Where $\theta: X \longrightarrow X''$ is as usual the James map defined by $\theta(x)(x^*)=\overline{x^*(x)}$.

(1.2) Suppose $|F|(E)=0$. Then $|F_x(E)|=|F(E)(x)|\le\|F(E)\|_{X'}\|x\|_X\le|F|(E)\|x\|_X$, $\forall x\in X$ $\Rightarrow |F_x(E)|=0$, therefore $F_x \ll |F|$, as claimed.

Now applying the usual Radon-Nikodym theorem for complex measures we obtain the existance of $f_x(\bullet)\in L^1(S,C)$ such that:

(*) $$F(E)x=\int_E f_x(s)d|F|,\ \forall E\in\Im$$

We also note that $|f_x(s)|\overset{\text{a.e.}}{\le}\|x\|_X$, because $|F|(E)\|x\|_X\ge\|F(E)\|_{X'}\|x\|_X\ge|F(E)x|=$

$$\overset{(*)}{=}\left|\int_E f_x(s)d|F|\right|,\text{ so } \|x\|_X \ge \frac{1}{|F|(E)}\left|\int_E f_x(s)d|F|\right|,\quad \forall E\in\Im \text{ (provided } |F|(E)\neq 0).$$

A well known theorem from real analysis implies that $|f_x(s)|\overset{\text{a.e.}}{\le}\|x\|_X$.

(2) On the next step we will be refining the map $f_x(\bullet)$ on a set of measure zero. Let $M_x=\{s\in S:\ |f_x(s)|>\|x\|_X\}$. Since X' is separable, so is X (by theorem 4). Let $D=\{x_i\}_{i=1}^{\infty}$ be a countable dense subset of X and $M_1=\bigcup_{i=1}^{\infty} M_{x_i}$. Then $|F|(M_1)\le\sum_{i=1}^{\infty}|F|(M_{x_i})=0$, since $|F|(M_{x_i})=0$, $\forall i\in N$, (recall that: $|f_x(s)|\overset{\text{a.e.}}{\le}\|x\|_X$).

Pick up $E\in\Im$, $a,b\in C$, $x,y\in D$, since $F(E)\in X'$,

$$\int_E f_{(ax+by)}(s)d|F|\overset{(*)}{=}F(E)(ax+by)=\bar{a}F(E)(x)+\bar{b}F(E)(y)=\int_E \bar{a}f_x(s)+\bar{b}f_y(s)d|F|,\quad \forall E\in\Im$$

This yields that $f_{ax+by}(s)\overset{\text{a.e.}}{=}\bar{a}f_x(s)+\bar{b}f_y(s)$. The exeptional set of measure zero depends on the choice of $a,b\in C$ and $x,y\in X$. Now, we want to refine $f_x(\bullet)$, so that the above equation holds for all $s\in S$.

Define $\tilde{D}=\left\{\sum_{k=1}^{m} a_k x_{i_k};\ \ a_k\in Q+iQ,\ x_{i_k}\in D,\ \ m<\infty\right\}$.

Then $\tilde{D}$ is a countable vector space over the field $(Q+iQ)$.

Let $z=\sum_{k=1}^{m} a_k x_{i_k}\in\tilde{D}$, $f_z(s)\overset{\text{a.e.}}{=}\sum_{k=1}^{m}\bar{a}_k f_{x_{i_k}}(s)$, denote by $M(z)$ the exeptional set, so $m(M(z))=0$. Also, let $M_2=\bigcup_{z\in\tilde{D}} M(z)$, thus $m(M_2)=0$.

Now, we have $f_z(s)=\sum_{k=1}^{m}\bar{a}_k f_{x_{i_k}}(s)$ and $|f_z(s)|\le\|z\|_X$, $\forall z\in\tilde{D}$, $s\notin M=M_1\cup M_2$.

Therefore, for all $s\notin M=M_1\cup M_2$ (note: $m(M)=0$), $a,b\in(Q+iQ)$ and $x,y\in\tilde{D}$ we have $f_{ax+by}(s)=\bar{a}f_x(s)+\bar{b}f_y(s)$.

(3) Define $h_x(s)=\begin{cases}\lim\limits_{z_n\to x} f_{z_n}(s), & \text{if } s\notin M\\ 0 & ,\ \text{if } s\in M\end{cases}$, where $x\in X$, $z_n\in\tilde{D}$, $z_n\xrightarrow[n\to\infty]{}x$ and $\tilde{D}$ is dense in X. Observe the following:

(3.1) $h_x(\bullet)$ is well defined. Suppose $z_n \xrightarrow[n\to\infty]{} x$, then
$\left|f_{z_n}(s)-f_{z_m}(s)\right|=\left|f_{z_n-z_m}(s)\right|\le\|z_n-z_m\|_X \xrightarrow[m,n\to\infty]{} 0$, so $\left\{f_{z_n}(s)\right\}_{n=1}^{\infty}$ is Cauchy.

Also, the limit is independent of the choice of the sequence $\{z_n\}_{n=1}^{\infty}\in\tilde{D}$, converging to x, because if $z_n \xrightarrow[n\to\infty]{} z$, $z'_n \xrightarrow[n\to\infty]{} z$, then $\left|f_{z'_n}(s)-f_{z_n}(s)\right|\le\|z'_n-z_n\|_X$. Thus, $f_z(s)=h_z(s)$ if $z\in\tilde{D}$ and $s\notin M$.

(3.2) Let $x\in X$ be arbitrary, $E\in\Im$ and $z_n \xrightarrow[n\to\infty]{} x$, $z_n\in\tilde{D}$. Remember that $|F|(S)<\infty$ and $\|x\|_X+1$ dominates $\left|f_{z_n}(s)\right|$ for every n large enough. By the Dominated Convergence theorem:

$$F(E)(x)=\lim_{n\to\infty}F(E)(z_n)=\lim_{n\to\infty}\int_E f_{z_n}(s)d|F|=\int_E h_x(s)d|F|.$$

If $s\notin M$, $|h_x(s)|=\lim_{n\to\infty}\left|f_{z_n}(s)\right|\le\lim_{n\to\infty}\|z_n\|_X=\|x\|_X$ while if $s\in M$, $|h_x(s)|=0\le\|x\|_X$.
Let $x,y\in X$, $a,b\in C$ and $x_n \xrightarrow[n\to\infty]{} x$, $y_n \xrightarrow[n\to\infty]{} y$, $a_n \xrightarrow[n\to\infty]{} a$, $b_n \xrightarrow[n\to\infty]{} b$, where $x_n,y_n\in\tilde{D}$, $a_n,b_n\in(Q+iQ)$, $\forall n\in N$. Then for $s\notin M$,
$h_{ax+by}(s)=\lim_{n\to\infty}f_{(a_nx_n+b_ny_n)}(s)=\lim_{n\to\infty}\left(\bar{a}_nf_{x_n}+\bar{b}_nf_{y_n}\right)=\bar{a}h_x(s)+\bar{b}h_y(s)$
while if $s\in M$, both sides equal zero. Thus $h_{ax+by}(s)=\bar{a}h_x(s)+\bar{b}h_y(s)$, $\forall s$.

Summarizing:

$$(**)\quad\left|\begin{array}{l} F(E)(x)=\int_E h_x(s)d|F| \\ |h_x(s)|\le\|x\|_X,\ \forall s\in S \\ h_{ax+by}(s)=\bar{a}h_x(s)+\bar{b}h_y(s),\ \forall s\in S \end{array}\right.$$

(4) Define $h(\bullet):S\longrightarrow X'$ by $h(s)x=h_x(s)$. Clearly, (**) implies $h(s)\in X'$, $\forall s\in S$. Moreover, $h(\bullet)\in L^1(S,X')$, because $h_x(\bullet)$ is a measurable scalar valued function,as a pointwise limit of measurable functions (namely $\chi_{M^c}(s)f_{z_n}(s)$, where $z_n \xrightarrow[n\to\infty]{} x$). Then, $h(\bullet)$ is weak-* measurable and

by Pettis theorem 3 $h(\bullet)$ is measurable (Re: X' is separable).

Also, $\|h(s)\|_{X'} = \sup_{\|x\|_X \le 1} |h(s)x| = \sup_{\|x\|_X \le 1} |h_x(s)| \overset{(**)}{\le} 1$, so $\|h\|_{L^1(S,X')} \le |F|(S) < \infty$.

Then (**) implies $F(E)(x) = \int_E h_x(s) d|F| = \int_E h(s)(x) d|F|$.

(5) Let $\Lambda \in X''$, then proposition 2 implies $\Lambda\left(\int_E h(s)d|F|\right) = \int_E \Lambda(h(s))d|F|, \ \forall E \in \Im$.

Hence, $\left(\int_E h(s)d|F|\right)(x) = \int_E h(s)(x)d|F|, \ \forall E \in \Im$ and therefore,

$$\left(\int_E h(s)d|F|\right)(x) = \int_E h(s)(x)d|F| = \int_E h_x(s)d|F| \overset{(**)}{=} F(E)(x), \ \forall E \in \Im, \ \forall x \in X$$

Then, $\int_E h(s)d|F| = F(E), \ \forall E \in \Im$.

(6) Finally, $F << m \Rightarrow |F| << m$. By the usual Radon-Nikodym theorem, there exists $k(\bullet) \in L^1(S,C)$, so that $|F|(E) = \int_E k(s)dm, \ \forall E \in \Im$. Thus the following claim implies: $F(E) = \int_E h(s)k(s)dm, \ \forall E \in \Im$.

Letting $g(s) = h(s)k(s), \ \forall s \in S$ wraps up the proof of this theorem.

⊗

Claim: Let $F(E) = \int_E h(s)d|F|$ and $|F|(D) = \int_D k(s)dm, \ \forall \ E, D \in \Im$. Then $F(E) = \int_E h(s)k(s)dm$, $\forall \ E \in \Im$ (i.e. $d|F| = k(s)dm$), where the functions $h(s)$ and $k(s)$ are as defined in theorem 9 .

Proof: Since $h(\bullet)$ is Bochner integrable with respect to $|F|$ (thus $h(\bullet)$ is measurable), there exists a sequence of simple functions $h_n(\bullet)$, such that:

(1) $h_n(s) = \sum_{k=1}^{p} c_k \chi_{E_k^n}(s)$; (2) $h_n(s) \xrightarrow[n\to\infty]{\text{pointwise}} h(s)$.

We observe that without loss of generality $\|h_n(s)\|_X \le \|h(s)\|_X + 1, \ \forall s \in S$, otherwise we multiply $h_n(s)$ by the characteristic function of the set $G_n = \{s \in S : \|h_n(s)\|_X \le \|h(s)\|_X + 1\}$, as we did in the proof of theorem 6. Under this modification, the functions $h_n(s)$ still satisfy the above two conditions.

Now let $E \in \Im$, the Dominated Convergence Theorem yields:

$$F(E) = \lim_{n\to\infty} \int_E h_n(s)d|F| = \lim_{n\to\infty} \int_E \sum_{k=1}^{p} c_k \chi_{E_k^n}(s)d|F| =$$

$$= \lim_{n\to\infty} \int_S \sum_{k=1}^{p} c_k \chi_{E_k^n \cap E}(s)d|F| = \lim_{n\to\infty} \sum_{k=1}^{p} c_k |F|\left(E_k^n \cap E\right) = \lim_{n\to\infty} \sum_{k=1}^{p} c_k \int_{E_k^n \cap E} k(s)dm =$$

$$= \lim_{n\to\infty} \sum_{k=1}^{p} c_k \int_S k(s)\chi_{E_k^n \cap E}(s)dm = \lim_{n\to\infty} \int_S \left(\sum_{k=1}^{p} c_k \chi_{E_k^n \cap E}(s) \right) k(s)dm = \lim_{n\to\infty} \int_S h_n(s)\chi_E(s)k(s)dm.$$

Because $\|h_n(s)\|_X \le \|h(s)\|_X + 1 \le 2$, so $\|h_n(s)k(s)\|_X \le 2k(s)$ (since $k(s) \ge 0$). Furthermore, $k(s) \in L^1(S,m) \Rightarrow 2k(s) \in L^1(S,m)$. Therefore, applying the Dominated Convergence theorem is justified (for real valued functions).

$$\lim_{n\to\infty} \int_S h_n(s)\chi_E(s)k(s)dm = \int_S \lim_{n\to\infty} h_n(s)\chi_E(s)k(s)dm = \int_S h(s)\chi_E(s)k(s)dm = \int_E h(s)k(s)dm.$$

Hence, $F(E) = \int_E h(s)k(s)dm$, $\forall E \in \Im$, as we claimed.

⊗

Corollary 3: If $(X, \| \|_X)$ is a separable, reflexive Banach space, then it has the Radon-Nikodym Property.

Proof: Let X be reflexive and separable. Then X'' is separable, since $X \cong X''$. However, $X'' = (X')'$ and so X'' is a separable anti-dual space. By the Radon-Nikodym Theorem (thm. 9) X'' has the Radon-Nikodym Property and therefore so does X.

⊗

Remark: In fact, it can be shown that any reflexive Banach space has the Radon-Nikodym Property (see "Vector Measures", Diestel & Uhl).

Definition 12: If X and Y are Banach spaces, we say that X is isometric to Y if there exists $\psi \in \mathcal{L}(X,Y)$ (i.e. $\psi : X \xrightarrow[\text{continuous}]{\text{linear}} Y$) such that $\psi(\bullet)$ is 1-1,onto and $\|\psi(x)\|_Y = \|x\|_X$, $\forall x \in X$. We write $X \cong Y$ to indicate, that X and Y are isometric. The map $\psi(\bullet)$ is called an isometry.

Our next goal is to show that, if X is a Banach space, X' has Radon-Nikodym Property and $(S,\Im,m)$ is a finite measure space, then $\left(L^p(S,X)\right)' \cong L^{p'}(S,X')$. To do this, we need some preliminary results.

Theorem 10: Let X be a Banach space and $(S,\Im,m)$ be a finite measure space. Let $p \geq 1$ and $\frac{1}{p}+\frac{1}{p'}=1$ (if $p=1 \Rightarrow p'=\infty$). Then $L^{p'}(S,X')$ is isometric to a subspace of $\left(L^p(S,X)\right)'$. Also, for $g(\bullet) \in L^{p'}(S,X')$,

(***) $$\sup_{\|f\|_{L^p(S,X)} \leq 1} \left| \int_S \langle g(s), f(s) \rangle dm \right| = \|g\|_{L^{p'}(S,X')}, \quad \text{where } \langle g,f \rangle = g(f).$$

Proof: (1) Suppose $f(\bullet)\in L^p(S,X)$ and $g(\bullet)\in L^{p'}(S,X')$. We claim, that $\langle g(\bullet),f(\bullet)\rangle\in L^1(S,C)$:

(1.1) Clearly, $\langle g(\bullet),f(\bullet)\rangle$ is measurable, since both maps are measurable;

(1.2) $\|\langle g(\bullet),f(\bullet)\rangle\|_{L^1(S,C)}=\int_S |\langle g(s),f(s)\rangle| dm=\int_S |g(s)(f(s))| dm\le\int_S \|g(s)\|_{X'}\|f(s)\|_X dm\le$

$$\le\left(\int_S \|g(s)\|_{X'}^{p'}dm\right)^{\frac{1}{p'}}\left(\int_S \|f(s)\|_X^{p}dm\right)^{\frac{1}{p}}=\|g\|_{L^{p'}(S,X')}\|f\|_{L^p(S,X)}<\infty.$$ Here, we used Holder's inequality.

(2) Define: $\psi: L^{p'}(S,X')\longrightarrow\left(L^p(S,X)\right)'$, by $\psi(g)(f)=\int_S \langle g(s),f(s)\rangle dm,\ \forall f(\bullet)\in L^p(S,X)$.

We note that ψ is linear and continuous and $\psi(g)$ is in $\left(L^p(S,X)\right)'$:

(2.1) $\psi(\lambda g_1+g_2)(f)=\int_S \langle \lambda g_1(s)+g_2(s),f(s)\rangle dm=\int_S (\lambda g_1(s)+g_2(s))(f(s))dm=$

$$=\int_S (\lambda g_1(s))(f(s))dm+\int_S (g_2(s))(f(s))dm=$$

$$=\int_S \lambda\langle g_1(s),f(s)\rangle dm+\int_S \langle g_2(s),f(s)\rangle dm=\lambda\psi(g_1)(f)+\psi(g_2)(f);$$

(2.2) $\|\psi g\|_{\left(L^p(S,X)\right)'}=\sup_{\|f\|_{L^p(S,X)}\le 1}|\psi(g)(f)|=\sup_{\|f\|_{L^p(S,X)}\le 1}\left|\int_S \langle g(s),f(s)\rangle dm\right|\le$

$$\le\sup_{\|f\|_{L^p(S,X)}\le 1}\int_S |\langle g(s),f(s)\rangle| dm\overset{(1.2)}{\le}\sup_{\|f\|_{L^p(S,X)}\le 1}\|g\|_{L^{p'}(S,X')}\|f\|_{L^p(S,X)}\le\|g\|_{L^{p'}(S,X')},$$

hence $\|\psi g\|_{\left(L^p(S,X)\right)'}\le\|g\|_{L^{p'}(S,X')}$ (i.e. $\psi(\bullet)$ is a bounded linear operator) and the image of ψ forms a subspace of $\left(L^p(S,X)\right)'$;

(2.3) $\psi(g)(\alpha f)=\int_S \langle g(s),\alpha f(s)\rangle dm=\int_S \overline{\alpha}\langle g(s),f(s)\rangle dm=\overline{\alpha}\psi(g)(f)$.

(3) To verify the second part, we need to show that $\|\psi g\|_{\left(L^p(S,X)\right)'}=\|g\|_{L^{p'}(S,X')}$ {for all $g(\bullet)\in L^{p'}(S,X')$ with $\|g\|_{L^{p'}(S,X')}>0$, otherwise it is obvious}. If so, equality (***) is established.

(3.1) First, let $g(s)=\sum_{k=1}^{p} c_k \chi_{E_k}(s)$ be simple, where $\{c_k\}_{k=1}^{p} \in X'$ and $\{E_k\}_{k=1}^{p}$ is a partition of S (i.e. $\{E_k\}_{k=1}^{p}$ – disjoint, $\bigcup_{k=1}^{p} E_k = S$).

It is clear, that $\|g(\bullet)\|_{X'} \in L^{p'}(S,C)$, since $g(\bullet) \in L^{p'}(S,X')$.

Let $\varepsilon > 0$ be given. Choose $h(\bullet) \in L^p(S,[0,\infty))$ such that:

$$h(s) \geq 0,\ \int_S \|g(s)\|_{X'} h(s)dm = \left(\int_S \|g(s)\|_{X'}^{p'} dm \right)^{\frac{1}{p'}} = \|g(s)\|_{L^{p'}(S,X')} \text{ and } \left(\int_S (h(s))^p dm \right)^{\frac{1}{p}} \leq 1.$$

(for example, let $h(s) = \|g(s)\|_{X'}^{(p'-1)} \alpha^{-1}$, where $\alpha = \left(\int_S \|g(s)\|_{X'}^{p'} dm \right)^{\frac{1}{p}}$).

Now, let $\{d_k\}_{k=1}^{p} \in X$ be chosen so that $\langle c_k, d_k \rangle \geq \|c_k\|_{X'} - \frac{\varepsilon}{\|h\|_{L^1(S,C)}}$, $\forall k = 1,\ldots,p$ and $\|d_k\|_X \leq 1$, $\forall k$. Also, let $f(s) = \sum_{k=1}^{p} d_k h(s) \chi_{E_k}(s)$. Then $f(\bullet) \in L^p(S,X)$ and $\|f(\bullet)\|_{L^p(S,X)} \leq 1$, by the choice of $h(\bullet)$ and $\|d_k\|_X \leq 1$, $\forall k$. Therefore,

$$\|\psi g\|_{(L^p(S,X))'} \geq |\psi(g)(f)| = \left| \int_S \langle g(s), f(s) \rangle dm \right| \geq \left| \int_S \sum_{k=1}^{p} \left(\|c_k\|_{X'} - \frac{\varepsilon}{\|h\|_{L^1(S,C)}} \right) h(s) \chi_{E_k}(s) dm \right| \geq$$

$$\geq \left| \int_S \|g(s)\|_{X'} h(s) dm \right| - \varepsilon \left| \int_S \frac{h(s)}{\|h\|_{L^1(S,C)}} dm \right| \geq \|g\|_{L^{p'}(S,X')} - \varepsilon.$$ Since $\varepsilon > 0$ was arbitrary small, $\|\psi g\|_{(L^p(S,X))'} \geq \|g\|_{L^{p'}(S,X')}$. Thus, by (2.2) we have the desired equality $\|\psi g\|_{(L^p(S,X))'} = \|g\|_{L^{p'}(S,X')}$, for simple functions.

(3.2) Finally, let $g(\bullet)$ be any function in $L^{p'}(S,X')$ and $\{g_n(\bullet)\}_{n=1}^{\infty}$ be a sequence of simple functions in $L^{p'}(S,X')$, such that $g_n \xrightarrow[n\to\infty]{} g$ in $L^{p'}(S,X')$. Because, ψ is linear and bounded, we get that

$$\|\psi g\|_{(L^p(S,X))'} = \lim_{n\to\infty} \|\psi g_n\|_{(L^p(S,X))'} \overset{(3)}{=} \lim_{n\to\infty} \|g_n\|_{L^{p'}(S,X')} = \|g\|_{L^{p'}(S,X')}.$$ This completes the proof of theorem 10. ⊗

Lemma 4: Suppose X is a Banach space, X' has the Radon-Nikodym Property, $(S,\Im,m)$ is a finite measure space and $l\in\left(L^p(S,X)\right)'$. Then $F(E)\in X'$, defined by $F(E)(x)=l(\chi_E(\bullet)x)$, $\forall x\in X$, is a vector measure with finite total variation (i.e. $|F|(S)<\infty$). Also, $F << m$.

Proof: (1) First we show that $F(E)$ is in X', the way it is defined.

Let $x,y\in X$, $\lambda\in C$, then $F(E)(\lambda x+y)=l(\chi_E(\bullet)(\lambda x+y))=l(\lambda\chi_E(\bullet)(x)+\chi_E(\bullet)(y))=$
$=\bar{\lambda}l(\chi_E(\bullet)(x))+l(\chi_E(\bullet)(y))=\bar{\lambda}F(E)(x)+F(E)(y)$. So $F(E)$ is conjugate linear.

Furthermore, $\|F(E)\|_{X'}=\sup_{\|x\|_X\le 1}|F(E)(x)|=\sup_{\|x\|_X\le 1}\left|l(\chi_E(\bullet)(x))\right|\le$

$\le\|l\|_{(L^p(S,X))'}\sup_{\|x\|_X\le 1}\|\chi_E(\bullet)(x)\|_{L^p(S,X)}\le\|l\|_{(L^p(S,X))'}(m(E))^{\frac{1}{p}}<\infty$, since $l(\bullet)\in\left(L^p(S,X)\right)'$.

Hence, $F(E)$ is bounded linear operator on X (i.e. it is continuous).

(2) Now let $\{E_k\}_{k=1}^{\infty}$ be a disjoint sequence of sets in $\Im$ and $E_\infty=\bigcup_{k=1}^{\infty}E_k\in\Im$.

$$\left|F(E_\infty)(x)-\sum_{k=1}^{n}F(E_k)(x)\right|=\left|l\left(\chi_{E_\infty}(\bullet)(x)\right)-\sum_{k=1}^{n}l\left(\chi_{E_k}(\bullet)(x)\right)\right|=$$

$$=\left|l\left(\chi_{E_\infty}(\bullet)(x)\right)-l\left(\sum_{k=1}^{n}\left(\chi_{E_k}(\bullet)(x)\right)\right)\right|\le\|l\|_{(L^p(S,X))'}\left\|\chi_{E_\infty}(\bullet)(x)-\sum_{k=1}^{n}\chi_{E_k}(\bullet)(x)\right\|_X\le$$

$$\le\|l\|_{(L^p(S,X))'}\left\|\chi_{E_\infty}(\bullet)(x)-\chi_{\bigcup_{k=1}^{n}E_k}(\bullet)(x)\right\|_{L^p(S,X)}\le\|l\|_{(L^p(S,X))'}m\left(\bigcup_{k\ge n+1}E_k\right)^{\frac{1}{p}}\|x\|_X.$$

Recall, that $m(S)<\infty$ and if $A_n=\bigcup_{k\ge n}E_k$, $\forall n\in N$, then $A_n\underset{n\to\infty}{\downarrow}\varnothing$, $m(A_1)<\infty$.

Since $m:\Im\xrightarrow[\text{measure}]{\text{positive}}[0,m(S)]$, we have $m(A_n)\underset{n\to\infty}{\downarrow}m(\varnothing)=0$

(i.e. $m\left(\bigcup_{k\ge n+1}E_k\right)\xrightarrow[n\to\infty]{}0$). Therefore, $\sum_{k=1}^{n}F(E_k)\xrightarrow[n\to\infty]{\text{in } X'}F(E_\infty)$.

So, F is a vector measure, as claimed. Furthermore, it is staightforward to see that $F << m$, using only the definition.

(3) To show that F has finite total variation (i.e. $|F|(S)<\infty$), we let $\{H_k\}_{k=1}^n \in \Im$ be any partition of S (i.e. $\{H_k\}_{k=1}^n$-disjoint and $\bigcup_{k=1}^n H_k = S$). Also, let $\varepsilon>0$ be given and $\|x_k\|_X \le 1$, $k=1,\dots,n$ be such that $\|F(H_k)\|_{X'} < F(H_k)(x_k)+\frac{\varepsilon}{n}$ (observe that $\left\|e^{i\tau}x_k\right\|_X = \|x_k\|_X$ and for an appropriate τ, $F(H_k)(e^{i\tau}x_k) = e^{-i\tau}F(H_k)(x_k)$ will be real).

Summing up for $k=1,\dots,n$, we get:

$$\sum_{k=1}^{n}\|F(H_k)\|_{X'} \le \sum_{k=1}^{n} F(H_k)(x_k)+\varepsilon = \sum_{k=1}^{n} l\left(\chi_{H_k}(\bullet)(x_k)\right)+\varepsilon \le$$

$$\le \|l\|_{(L^p(S,X))'}\left\|\sum_{k=1}^{n}\chi_{H_k}(\bullet)(x_k)\right\|_{L^p(S,X)} + \varepsilon = \|l\|_{(L^p(S,X))'}\left(\int_S\left\|\sum_{k=1}^{n}\chi_{H_k}(s)(x_k)\right\|_X^p dm\right)^{\frac{1}{p}} + \varepsilon \le$$

$$\le \|l\|_{(L^p(S,X))'}\left(\int_S\left|\sum_{k=1}^{n}\chi_{H_k}(s)\right|^p dm\right)^{\frac{1}{p}} + \varepsilon = \|l\|_{(L^p(S,X))'}(m(S))^{\frac{1}{p}} + \varepsilon < \infty .$$

Then $|F|(S)<\infty$, since $\{H_k\}_{k=1}^n$ is an arbitrary partition of S (recall: $|F|(S) = \sup\left\{\sum_{k=1}^{n}\|F(H_k)\|_{X'} : \{H_k\}_{k=1}^n - \text{disjoint partition of } S\right\}$).

⊗

Now we will state and prove the Riesz-Representation Theorem for L^p spaces of vector valued functions.

Theorem 11: Let X be a Banach space, X' has the Radon-Nikodym Property and $(S,\Im,m)$ be a finite measure space. Then $\left(L^p(S,X)\right)' \cong L^{p'}(S,X')$ (we will be showing that the map ψ, from theorem 10, is onto).

Proof: (1) Suppose $l \in \left(L^p(S,X)\right)'$. We define a vector measure $F(E) \in X'$ by $F(E)(x) \overset{(a)}{=} l(\chi_E(\bullet)(x)),\ \forall x \in X$. Then, lemma 4 yields that F is absolutely continuous with respect to m (i.e. $F \ll m$) and has finite total variation (i.e. $|F|(S) < \infty$). Furthermore, since X' has the Radon-Nikodym Property, there exists $g(\bullet) \in L^1(S,X')$ so that

$$F(E) \overset{(b)}{=} \int_E g(s)dm,\ \forall E \in \Im.$$

(2) Now, let $h(s) = \sum_{k=1}^{n} c_k \chi_{E_k}(s)$ be a simple function. Then

$$l(h) = l\left(\sum_{k=1}^{n} c_k \chi_{E_k}\right) = \sum_{k=1}^{n} l\left(c_k \chi_{E_k}\right) \overset{\text{by }(a)}{=} \sum_{k=1}^{n} F(E_k)(c_k) \overset{\text{by }(b)}{=}$$

$$= \sum_{k=1}^{n} \int_{E_k} \langle g(s)\ ,\ c_k \rangle dm = \int_S \left\langle g(s)\ ,\ \sum_{k=1}^{n} \chi_{E_k}(s) c_k \right\rangle dm = \int_S \langle g(s)\ ,\ h(s) \rangle dm\ .$$

Here we used that $g(\bullet) \in L^1(S,X')$, (i.e. $g(\bullet)$ is conjugate linear on X).

If we knew $g(\bullet) \in L^{p'}(S,X')$, then this would mean $\psi(g) = l$, because the simple functions are dense in $L^p(S,X)$. However, we only know $g(\bullet) \in L^1(S,X')$. Now, we claim that in fact $g(\bullet) \in L^{p'}(S,X')$.

(3) Let $G_n = \left\{ s \in S:\ \|g(s)\|_{X'} \le n \right\}$. Define $\tilde{l}(h) = l\left(h\chi_{G_n}\right)$. Then $\tilde{l} \in \left(L^p(S,X)\right)'$ and for any $h(\bullet)$ simple in $L^p(S,X)$

$$\tilde{l}(h) = l(h\chi_{G_n}) = \int_S \left\langle g(s)\ ,\ \left(h\chi_{G_n}\right)(s) \right\rangle dm = \int_S \left\langle \chi_{G_n}(s) g(s)\ ,\ h(s) \right\rangle dm$$

Since simple functions are dense and $\left(\chi_{G_n} g\right)(\bullet) \in L^{p'}(S,X')$, this equation holds for all $h(\bullet) \in L^p(S,X)$. Therefore, by theorem 10,

$$\left\| g\chi_{G_n} \right\|_{L^{p'}(S,X')} = \sup_{\|h\|_{L^p(S,X)} \le 1} \left| \int_S \left\langle \left(\chi_{G_n} g\right)(s)\ ,\ h(s) \right\rangle dm \right| = \sup_{\|h\|_{L^p(S,X)} \le 1} \left| l\left(h\chi_{G_n}\right) \right| \le \|l\|_{\left(L^p(S,X)\right)'}.$$

Therefore, by the Monotone Convergence Theorem,

$$\|g\|_{L^{p'}(S,X')} \leq \|l\|_{(L^p(S,X))'} .$$

It follows that $g(\bullet) \in L^{p'}(S,X')$. Also, the series of inequalities in (2) hold for all $h(\bullet) \in L^p(S,X)$. Hence

$$l = \psi g$$

and ψ is onto. Then, theorem 10 yields $L^{p'}(S,X') \cong \left(L^p(S,X)\right)'$.

⊗

The next two corollaries identify classes of Banach spaces $(X, \|\ \|_X)$ for which we can characterize the anti-dual space of $L^p(S,X)$ for $p \geq 1$.

<u>Corollary 4:</u> If X' is a separable anti-dual space, then $L^{p'}(S,X') \cong \left(L^p(S,X)\right)'$.

<u>Proof:</u> Separability of X' implies that X' has the Radon-Nikodym Property (by theorem 9). $(S,\Im,m)$ is assumed to be a finite measure space, therefore, theorem 11 yields $L^{p'}(S,X') \cong \left(L^p(S,X)\right)'$.

⊗

<u>Corollary 5:</u> If X is separable and reflexive, then $L^{p'}(S,X') \cong \left(L^p(S,X)\right)'$.

<u>Proof:</u> By corollary 2, X' is separable, since X is separable and reflexive. Applying the preceding corollary 4 wraps up the proof.

⊗

Corollary 6: Let $(S,\Im,m)$ be a finite measure space and $(X,\|\ \|_X)$ be a separable reflexive Banach space. Then $L^p(S,X)$ is reflexive for $1<p<\infty$.

Remark: It can be shown, that $L^1(S,X)$ is separable and $\left(L^1(S,X)\right)' = L^\infty(S,X')$. However, $L^\infty(S,X')$ is not always separable itself. Therefore, by the converse of corollary 2, $L^\infty(S,X')$ and $L^1(S,X)$ are not always reflexive.

Proof: (1) First, by theorems 10 and 11 the map $\psi:\ L^{p'}(S,X')\longrightarrow\left(L^p(S,X)\right)'$, defined by , $\psi(f^*)(f)=\int_S\left\langle f^*(s)\ ,\ f(s)\right\rangle dm\ \ \forall\ f(\bullet)\in L^p(S,X)$, is linear, 1-1, onto and continuous. Also, same holds for the inverse map $\psi^{-1}:\ \left(L^p(S,X)\right)'\longrightarrow L^{p'}(S,X')$. We have the following integral equation for ψ^{-1}:

$$\int_S\left\langle\left(\psi^{-1}(l)\right)(s)\ ,\ f(s)\right\rangle dm=\left\langle\psi\left(\psi^{-1}(l)\right)\ ,\ f\right\rangle=\langle l\ ,\ f\rangle,$$

where:

$$\left|\begin{array}{l} f(\bullet)\in L^p(S,X)\\ f^*(\bullet)\in L^{p'}(S,X')\\ f^{**}(\bullet)\in L^p(S,X'')\end{array}\right.\qquad\left|\begin{array}{l} l\in\left(L^p(S,X)\right)'\\ h\in\left(L^{p'}(S,X')\right)'\end{array}\right.$$

From now on, a new subindex of the mapping ψ comes into play, showing exactly which L^p space ψ acts on.

(2) Define:

$$\left|\begin{array}{l}\psi_p:L^{p'}(S,X')\longrightarrow\left(L^p(S,X)\right)'\\ \psi_{p'}:L^p(S,X'')\longrightarrow\left(L^{p'}(S,X')\right)'\end{array}\right.\quad\text{by}\quad\left|\begin{array}{l}\left\langle\psi_pf^*\ ,\ f\right\rangle=\int_S\left\langle f^*(s)\ ,\ f(s)\right\rangle dm\\ \left\langle\psi_{p'}f^{**}\ ,\ f^*\right\rangle=\int_S\left\langle f^{**}(s)\ ,\ f^*(s)\right\rangle dm\end{array}\right.$$

Respectively for the inverse mappings:

$$\left|\begin{array}{l}\psi_p^{-1}:\left(L^p(S,X)\right)'\longrightarrow L^{p'}(S,X')\\ \psi_{p'}^{-1}:\left(L^{p'}(S,X')\right)'\longrightarrow L^p(S,X'')\end{array}\right.\quad\text{and}\quad\left|\begin{array}{l}\int_S\left\langle\left(\psi_p^{-1}(l)\right)(s)\ ,\ f(s)\right\rangle dm=\langle l\ ,\ f\rangle\\ \int_S\left\langle\left(\psi_{p'}^{-1}(h)\right)(s)\ ,\ f^*(s)\right\rangle dm=\left\langle h\ ,\ f^*\right\rangle\end{array}\right..$$

(3) Let $\theta : X \longrightarrow X''$ and $\Theta : L^p(S,X) \longrightarrow \left(L^p(S,X)\right)''$ be the James maps $\left\langle \theta(x), x^* \right\rangle = \overline{\left\langle x^*, x \right\rangle}$, $\langle \Theta(f), l \rangle = \overline{\langle l, f \rangle}$. We will be showing that Θ is onto (i.e. that $L^p(S,X)$ is reflexive).

Define $\hat{\theta} : L^p(S,X) \longrightarrow L^p(S,X'')$ by $\left(\hat{\theta}(f)\right)(s) = \theta(f(s))$. Then $\hat{\theta}$ is onto since X is reflexive (i.e. θ is onto).

Consider the following diagram:

$$\begin{array}{ccc} \left(L^p(S,X)\right)'' & \xleftarrow{\left(\psi_p^{-1}\right)^*} & \left(L^{p'}(S,X')\right)' \\ \Theta \uparrow & & \uparrow \psi_{p'} \\ L^p(S,X) & \xrightarrow{\hat{\theta}} & L^p(S,X'') \end{array}$$

By the Riesz Representation Theorem (theorem 11), we know that $\psi_{p'}$ and $\left(\psi_p^{-1}\right)^*$ are both onto mappings, where $\left(\psi_p^{-1}\right)^*$ is defined as follows:

$$\begin{array}{ccc} \left(L^p(S,X)\right)'' & \xleftarrow{\left(\psi_p^{-1}\right)^*} & \left(L^{p'}(S,X')\right)' \\ \left(L^p(S,X)\right)' & \xrightarrow{\psi_p^{-1}} & L^{p'}(S,X') \end{array}, \qquad \left\langle \left(\psi_p^{-1}\right)^*(h) \, , \, l \right\rangle = \left\langle h \, , \, \psi_p^{-1}(l) \right\rangle.$$

(4) Finally, we observe that $\Theta = \left(\left(\psi_p^{-1}\right)^* \circ \psi_{p'} \circ \hat{\theta} \right)$.

$$\left\langle \left(\left(\psi_p^{-1}\right)^* \circ \psi_{p'} \circ \hat{\theta} \right)(f) \, , \, l \right\rangle = \left\langle \left(\psi_{p'} \circ \hat{\theta}\right)(f) \, , \, \psi_p^{-1}(l) \right\rangle = \int_S \left\langle \left(\hat{\theta} f\right)(s) \, , \, \left(\psi_p^{-1}(l)\right)(s) \right\rangle dm =$$

$$= \int_S \left\langle \theta(f(s)) \, , \, \left(\psi_p^{-1}(l)\right)(s) \right\rangle dm = \int_S \overline{\left\langle \left(\psi_p^{-1}(l)\right)(s) \, , \, f(s) \right\rangle} dm = \overline{\langle l \, , \, f \rangle} = \langle \Theta(f) \, , \, l \rangle.$$

Hence, Θ is onto because $\Theta = \left(\left(\psi_p^{-1}\right)^* \circ \psi_{p'} \circ \hat{\theta} \right)$ and $\left(\psi_p^{-1}\right)^* \circ \psi_{p'} \circ \hat{\theta}$ is onto. This proves the corollary.

⊗

Appendix

[1] Theorem: {Inequalities of Holder and Minkowski} Suppose $1 \le p, p' \le \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$ (if $p = 1 \Rightarrow p' = \infty$). Then for f and g measurable real valued functions the following two inequalities hold:

(1) $\int_S |f||g| dm \le \left(\int_S |f|^p dm \right)^{\frac{1}{p}} \left(\int_S |g|^{p'} dm \right)^{\frac{1}{p'}}$;

(2) $\|f + g\|_p = \left(\int_S |f + g|^p dm \right)^{\frac{1}{p}} \le \left(\int_S |f|^p dm \right)^{\frac{1}{p}} + \left(\int_S |g|^p dm \right)^{\frac{1}{p}} = \|f\|_p + \|g\|_p$.

[2] Theorem: {Hahn-Banach} Let M be a subspace of a complex vector space X, and suppose $f : M \xrightarrow[\text{linear}]{\text{conjugate}} C$ and $|f(x)| \le K\|x\|_X$ for all $x \in M$, where K is a const. Then there exists a conjudate linear function $\tilde{f}$ (extention of f) so that $\tilde{f}/_M = f$ and $|\tilde{f}(x)| \le K\|x\|_X$, $\forall\ x \in X$.

[3] Corollary 1: {Separation theorem} Suppose X is a normed vector space and K is a closed convex subset of X. If $p \in K^c$ $(= X - K)$, then there exists a real number r, such that $\operatorname{Re} f(p) > r > \operatorname{Re} f(k)$, $\forall\ k \in K$.

[4] Corollary 2: Let X be a normed vector space and K be a closed subspace of X. If $p \in K^c$ $(= X - K)$, then there exists $f \in X'$, such that $f(k) = 0$, $\forall\ k \in K$ and $f(p) \ne 0$.

[5] Theorem: {Monotone Convergence Th. for positive real valued functions} Let $(S, \Im, m)$ be a positive measure space and $\{f_n\}_{n=1}^{\infty}$ be a sequence of measurable functions such that $f_n \le f_{n+1}$, $\lim_{n \to \infty} f_n(s) = f(s)$, $\forall\ s \in S$ and $f_n(s), f(s) \ge 0$, $\forall\ s \in S$. Then f is measurable and $\lim_{n \to \infty} \int_S f_n(s) dm = \int_S f(s) dm$.

[6] **Theorem:** {Fatou's Lemma} Let $(S,\Im,m)$ be a positive measure space and $\{f_n\}_{n=1}^{\infty}$ be a sequence of measurable functions such that $f_n(s)\geq 0,\ \forall\ s\in S$. If $g(s)=\liminf_{n\to\infty} f_n(s)$, then g is measurable and $\liminf_{n\to\infty}\int_E f_n(s)dm \geq \int_E g(s)dm,\ \forall E\in\Im$.

[7] **Theorem:** {Dominated Convergence Theorem for real valued functions} Let $\{f_n\}_{n=1}^{\infty}$ be a convergent sequence of measurable functions and $\lim_{n\to\infty} f_n(s)=f(s)$. Then if there exists a measurable function $g\geq 0$, so that $|f_n(s)|\leq g(s),\ \forall\ n\in N,\ \forall\ s\in S$ and $\int_S g(s)dm<\infty$ we have that f is measurable and $\lim_{n\to\infty}\int_S f_n(s)dm=\int_S f(s)dm<\infty$.

[8] **Theorem:** {Riesz Representation Theorem} Let $p>1$, $(S,\Im,m)$ be a finite measure space and $\Lambda\in\left(L^p(S,m)\right)'$. Then there exists a unique $h\in L^{p'}(S,m)$, so that $\Lambda(f)=\int_S h(s)\overline{f(s)}dm,\ \forall\ f\in L^p(S,m)$, where $\frac{1}{p}+\frac{1}{p'}=1$.

[9] **Theorem:** {Radon-Nikodym Theorem for finite positive measures} Let λ and μ be positive measures defined on on a measure space $(S,\Im)$. Suppose λ is absolutely continuous with respect to μ (i.e. $\lambda<<\mu$, $\mu(E)=0\Rightarrow\lambda(E)=0$). Then, there exists $f\in L^1(S,\mu)$, such that $f(s)\geq 0$ and $\lambda(E)=\int_E f(s)d\mu,\ \forall\ E\in\Im$.

[10] **Theorem:** {Fubini} Let $f:X\times Y\longrightarrow[0,\infty]$ be measurable with respect to the σ-algebra $\Im\times\zeta$. Then

$$\int_{X\times Y} fd(\lambda\times\mu)=\int_X\int_Y f(x,y)d\mu d\lambda=\int_Y\int_X f(x,y)d\lambda d\mu\ .$$

[11] **Theorem:** {Eberlein Smulian} If $(Y,\|\ \|_Y)$ is a reflexive Banach space, then the unit ball in Y is weakly sequentially compact (i.e. if $\{x_k\}_{k=1}^{\infty}\in B(0,1)$, then $\{x_{k_n}\}_{n=1}^{\infty}\subset\{x_k\}_{k=1}^{\infty}:\ x_{k_n}\xrightarrow[n\to\infty]{\text{Weakly}}x\in B(0,1)$).

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